UNIQUENESS OF THE INVARIANT MEAN ON ABELIAN TOPOLOGICAL SEMIGROUPS

BY

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Let S be a topological semigroup[1] and let $C(S)$ be the space of bounded continuous real-valued functions x on S with

$$\|x\| = \operatorname*{Sup}_{\sigma \in S} | x(\sigma) | .$$

For each $\sigma \in S$ we define the *left translation operator*

$$l_\sigma : C(S) \to C(S)$$

by

$$(l_\sigma x)(\tau) = x(\sigma\tau)$$

and the *right translation operator*

$$r_\sigma : C(S) \to C(S)$$

by

$$(r_\sigma x)(\tau) = x(\tau\sigma).$$

An element $\nu \in C(S)^*$ is called a *mean* if

$$\|\nu\| = 1 = \nu(e)$$

where e is the function which is identically one. An element ν in $C(S)^*$ is called left-invariant if $l_\sigma^* \nu = \nu$ for every σ in S and is called right-invariant if $r_\sigma^* \nu = \nu$ for every $\sigma \in S$. We say that ν is invariant if $l_\sigma^* \nu = \nu = r_\sigma^* \nu$ for every σ in S. S is called amenable if there exists an invariant mean. It is known that an Abelian topological semigroup is amenable.

In my earlier paper [2] I proved that a discrete Abelian semigroup has a unique invariant mean if and only if it has a finite ideal. It is quite reasonable to conjecture that in general an Abelian topological semigroup has a unique invariant mean if and only if the semigroup has a compact ideal. In this paper we prove the conjecture in certain special situations.

THEOREM 1[2]. *An abelian topological semigroup with a compact ideal has a unique invariant mean.*

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(1) By a topological semigroup we mean a semigroup provided with a Hausdorff topology in which the mapping $(\sigma, \tau) \to \sigma\tau$ of $S \times S$ into S is continuous.

(2) In the original manuscript this theorem was proved under the assumption that the semigroup is normal. The author is grateful to the referee for suggesting a method for removing the condition of normality.

Proof. Let Δ be a compact ideal of S. If $\Delta_1, \cdots, \Delta_n$ be closed ideals of S, each contained in Δ, then

$$\Delta^* = \bigcap_{i=1}^{n} \Delta_i \supset \Delta_1 \cdots \Delta_n$$

and so is nonempty. Thus the family $\mathfrak{F}$ of all closed ideals of S contained in Δ has the finite intersection property and so due to the compactness of Δ

$$A = \bigcap_{\Delta' \in \mathfrak{F}} \Delta' \neq \emptyset.$$

Thus A is an ideal. This A is obviously a minimal compact ideal of S. Take any $a \in A$. Then aA is an ideal contained in A and so $aA = A$ due to the minimal character of A. Therefore A is a group.

We now define a relation among the elements of A. We say that two elements a and a' in A are equivalent if they cannot be separated by a continuous function on S, i.e., if $x(a) = x(a')$ for every x in $C(S)$. This is obviously an equivalence relation. Let H be the set of elements of A which are equivalent to the identity e of the group A. We claim that H is a closed subgroup of A and that the equivalence classes are simply the cosets of H in A. That H is closed follows from the fact that

$$H = \bigcap_{x \in C(S)} \{a \mid a \in A, x(a) = x(e)\}.$$

To prove the remaining assertion we proceed as follows:

For each $\sigma \in S$, we define $\hat{\sigma}$ by

$$\hat{\sigma}(s) = \sigma s; \; s \in S.$$

If a and b are equivalent then ab^{-1} and e are equivalent, for

$$x(ab^{-1}) = (x \circ (b^{-1})\hat{\ })(a) = (x \circ (b^{-1})\hat{\ })(b) = x(e).$$

Conversely the equivalence of ab^{-1} and e implies the equivalence of a and b, since

$$x(a) = (x \circ \hat{b})(ab^{-1}) = (x \circ \hat{b})(e) = x(b).$$

From this can be easily deduced the assertions made above.

We now form the quotient group A/H. Each x in $C(S)$ defines an $\tilde{x}$ in $C(A/H)$. It is obvious that if $\tilde{x}_1 = \tilde{x}_2$ then x_1 and x_2 agree on A.

The set $\tilde{C} = \{\tilde{x} \mid x \in C(S)\}$ separates the points of A/H. Since A/H is compact the set $\tilde{C}$ is dense in $C(A/H)$ by the Stone-Weierstrass Theorem.

Let ν be any invariant mean on $C(S)$. If z is in $C(S)$ we define

$$\|z\|_A = \operatorname{Sup}_{a \in A} |z(a)|.$$

If z_λ, $\lambda \in \Lambda$ is a net of elements of $C(S)$ such that $\|z_\lambda\|_A \to 0$ then $\nu(z_\lambda) \to 0$ for $|\nu(z_\lambda)| = |\nu(l_a z_\lambda)| \leqq \|l_a z_\lambda\| \leqq \|z_\lambda\|_A$ where a is any fixed element of A.

These facts allow us to define invariant integration on A/H as follows:

Let $y \in C(A/H)$. We can choose $\tilde{x}_n$ in $\tilde{C}$ such that $\tilde{x}_n \to y$. It is obvious that $\|x_m - x_n\|_A \to 0$ as $m, n \to \infty$ and so $\nu(x_m - x_n) \to 0$ as $m, n \to \infty$. Thus $\nu(x_n)$ is a Cauchy sequence and so possesses a unique limit. This limit depends only on y since for any other sequence $\tilde{z}_n$ converging to y we have

$$\|\tilde{x}_n - \tilde{z}_n\| \to 0.$$

Thus $\|x_n - z_n\|_A \to 0$ and so $\lim \nu(x_n) = \lim \nu(z_n)$. We define

$$\int_{A/H} y = \lim_n \nu(x_n).$$

That the integral is invariant follows from the fact that for any $\bar{a}$ in A/H, $\bar{l}_{\bar{a}}(\tilde{x}) = (l_a x)^{\sim}$ and that $\nu(l_a x) = \nu(x)$. Since A/H is a compact Abelian group the above integral coincides with the Haar integral on A/H. This proves the uniqueness of ν. Since S is an Abelian semigroup we know that there is at least one invariant mean on $m(S)$ and so also on $C(S)$. The proof of Theorem 1 is now complete.

THEOREM 2. *Let S and S' be Abelian topological semigroups and let f: $S \to S'$ be a continuous homomorphism of S onto S'. Let F: $C(S') \to C(S)$ be defined by*

$$(Fx')(\sigma) = x'(f(\sigma)),\ x' \in C(S'),\ \sigma \in S.$$

Then F^* carries the set of invariant means on $C(S)$ onto the set of invariant means on $C(S')$. Consequently the existence of many invariant means on $C(S')$ implies the existence of many invariant means on $C(S)$.

Proof. Let μ be an invariant mean on $C(S)$. Then $F^*\mu$ is a positive linear functional and since $(F^*\mu)(e') = \mu(Fe') = \mu(e) = 1$, we see that $F^*\mu$ is a mean on $C(S')$. Let $x' \in C(S')$ and let σ be an element of S. Then it can be easily seen that

$$l_\sigma(Fx') = F(l'_{f\sigma}x').$$

If $\sigma' \in S'$ we may take $\sigma \in S$ such that $f\sigma = \sigma'$. Then

$$\begin{aligned}(F^*\mu)(l'_{\sigma'}x') &= \mu[F(l'_{f\sigma}x')] = \mu[l_\sigma(Fx')] \\ &= \mu(Fx') = (F^*\mu)(x').\end{aligned}$$

Consequently $F^*\mu$ is an invariant mean on $C(S')$.

Suppose now that μ' is an invariant mean on $C(S')$. Let

$$C_0 = \{Fx' \mid x' \in C(S')\}$$

and define μ_0 on C_0 by

$$\mu_0(Fx') = \mu'(x').$$

μ_0 is well-defined since F is a 1-1 mapping. Since $l_\sigma(Fx') = F(l'_{f\sigma}x')$ we conclude that C_0 is invariant under every l_σ. We next observe that μ_0 is invariant on C_0 under all the operators l_σ. This follows from the following calculation:

$$\begin{aligned}\mu_0[l_\sigma(Fx')] &= \mu_0[F(l'_{f\sigma}x')] = \mu'(l'_{f\sigma}x') \\ &= \mu'(x') = \mu_0(Fx').\end{aligned}$$

Thus we have a linear functional μ_0 defined on an invariant subspace C_0 and invariant under all the operators l_σ. We now use a theorem of Silverman [3] to obtain an extension μ of μ_0 which is an invariant mean on $C(S)$. It can be easily verified that $F^*\mu = \mu'$.

It may be remarked that if S and S' are completely regular spaces in which the translations are uniformly continuous mappings (this condition is satisfied if S and S' are groups) and if $f: S \to S'$ is uniformly continuous from S onto S' then the above method can be used to prove that the existence of many invariant means on $UC(S')$, the space of bounded, real-valued uniformly continuous functions on S, implies the existence of many invariant means on $UC(S)$.

We say that an Abelian topological group G has the property P' if there exists a countable subgroup H and a symmetric neighborhood V of 0 such that

(i) $(V+V)\cap H = \{0\}$,

(ii) V is maximal among the symmetric neighborhoods of 0 which satisfy (i),

(iii) there exists a neighborhood W of 0 such that $V+V+W$ meets H at only finitely many points.

It may be observed that if G contains a countable discrete subgroup H we can easily find, by an application of Zorn's lemma, a symmetric neighborhood V of 0 which satisfies (i) and (ii). Thus (iii) is the strong condition.

An Abelian topological group will be said to have property P if G or a factor group of G has property P'. The property P is not very restrictive since many groups which can at all be expected to possess this property do possess it. Obviously an infinite discrete group G has the property P' with H any countable subgroup of G, V constructed by Zorn's lemma and with $W=0$. Next any subgroup $G \neq \{0\}$ of the additive group of real numbers has the property P'. Without loss of generality we assume that $1 \in G$ and we let H be the cyclic subgroup generated by 1. Then

$$V = \{x \mid -1/2 < x < 1/2, x \in G\} = W$$

satisfy (i), (ii) and (iii). Again any nonzero subgroup G of the additive group of a normed linear space X has property P. For let $0 \neq \alpha \in G$. We can find a linear functional f such that $f(\alpha) \neq 0$. Thus G would be mapped homomorphically onto a nonzero subgroup fG of the additive group of real numbers which, as we have shown, possesses P'. Thus G has P. Finally any locally compact Abelian group G which is not compact has the property P. To prove this we use the fact that G is isomorphic to $R_p \times G_1$ [4] where G_1 is a group which contains a compact subgroup H such that G_1/H is discrete. If $p \neq 0$ then R, the additive group of real numbers, is a homomorphic image of G and so G has property P since R has property P'. If $p=0$, G_1/H must be infinite since otherwise G would be compact. Since G_1/H has P' it follows that G has P.

Theorem 3. *Let G be an Abelian topological group having property P. Then there are many invariant means on $C(G)$.*

Proof. By virtue of Theorem 2 we may assume that G has P'. Thus there exists a symmetric neighborhood V of 0 and a countable subgroup H such that

(i) $(V+V)\cap H=\{0\}$,

(ii) V is maximal among the symmetric neighborhoods of 0 which satisfy (i),

(iii) there exists a neighborhood W such that $(V+V+W)\cap H$ is a finite set.

We now take up the proof in several steps.

(a) It is obvious that if $v+h=v'+h'$, v, $v'\in V$; h, $h'\in H$, then $v=v'$ and $h=h'$. This is a consequence of (i).

(b) If g is not in

$$\overline{V+H}$$

then $0\neq 2g\in H$. To prove this we choose W', a symmetric neighborhood of 0, such that $(g+W')\cap(H+V)=\emptyset$ and $W'+W'\subset V$. Then

$$V' = (g+W')\cup V\cup(-g+W')$$

is a symmetric neighborhood of 0 which is larger than V. Consequently $V'+V'$ must meet H. It follows therefore that $2g=w_1'+w_2'+h=v+h$. By (a) this v is independent of W' and since $v\in W'+W'$ for every sufficiently small W', it follows that $v=0$ (the group topology is Hausdorff). Thus $2g=h$. If $h=0$, then

$$V^* = V\cup(g+W^*)$$

will contradict the maximal character of V where W^* is a symmetric neighborhood of 0 such that

$$(g+W^*)\subset(\overline{V+H})^c \quad\text{and}\quad W^*+W^*\subset V.$$

(c) Let G_2 be the set of elements $g\in G$ for which $2g=0$. We claim that either

$$\overline{V+H} = G$$

or G_2 is a neighborhood of 0. To prove this take

$$g \notin \overline{V+H}$$

and choose an open neighborhood U of 0 such that

$$(g+U) \cap (\overline{H+V}) = \emptyset, \qquad U+U \subset V.$$

Take any $u \in U$. Then

$$g+u \notin \overline{V+H}$$

and therefore $2g+2u=h$. Since $2g=h'$, it follows from (a) that $2u=0$. Thus G_2 contains U and our assertion is proved.

(d) We will now construct two subsets A and B of H which have the following properties:

(i) given any finite subset $\{h_1, \cdots, h_n\}$ of elements of H there exists an $h \in H$ such that $h_1+h, \cdots, h_n+h$ are all in A and an h' such that h_1+h', $\cdots, h_n+h'$ are in B.

(ii) $(A+V+W) \cap (B+V) = \emptyset$.

$(V+V+W) \cap H$ is finite. Let its members be $h^{(1)}, \cdots, h^{(l)}$. Let us first assume that H is finitely generated. Consequently, by the fundamental theorem on finitely generated Abelian groups, there exist $h_1, \cdots, h_t$ in H and a finite subgroup Φ of H such that every element h of H can be uniquely written in the form

$$\lambda_1 h_1 + \cdots + \lambda_t h_t + \phi; \qquad \lambda_i \text{ integers}, \phi \in \Phi.$$

We shall call $\lambda_1, \cdots, \lambda_t$ the co-ordinates of h. We define A_1 to be Φ. Suppose that finite subsets $A_1, A_2, \cdots, A_p$ of H have been defined. Let μ_p be an integer larger than the absolute value of every co-ordinate of every member of the finite set

$$\left\{h^{(i)} + h \,\middle|\, 1 \leqq i \leqq l,\ h \in \bigcup_{j=1}^{p} A_j\right\}$$

and define A_{p+1} by

$$A_{p+1} = \left\{h = \sum_{i=1}^{t} \lambda_i h_i + \phi \,\middle|\, \phi \in \Phi, \mu_p \leqq |\lambda_1|, \cdots, |\lambda_t| \leqq \mu_p + 10^p\right\}.$$

We let

$$A = \bigcup_{j=1}^{\infty} A_{2j-1} \quad \text{and} \quad B = \bigcup_{j=1}^{\infty} A_{2j}.$$

It can be easily verified that conditions (i) and (ii) are satisfied.

Suppose now that H is not finitely generated. Since H is countable, we enumerate the elements of H as $h^{(1)}, \cdots, h^{(r)}, \cdots$ where $h^{(1)}, \cdots, h^{(l)}$ are the members of H which are in $V+V+W$. We define sets $A_1, A_2, \cdots, A_p, \cdots$, not necessarily finite, in the following manner.

Let A_1 be the subgroup generated by $h^{(1)}, \cdots, h^{(l)}$. Suppose $A_1, A_2, \cdots, A_p$ have been defined in such a way that $\bigcup_{j=1}^{p} A_j$ is a finitely generated subgroup H_p of H. H_p being not equal to H, let h be the first among $h^{(1)}, \cdots, h^{(r)}, \cdots$ which is not in H_p and let A_{p+1} consist of those elements of the subgroup generated by H_p and h which are not in H_p.

We now define

$$A = \bigcup_{j=1}^{\infty} A_{2j-1}, \qquad B = \bigcup_{j=1}^{\infty} A_{2j}.$$

It can again be easily verified that A and B have the properties (i) and (ii).

(e) We now prove a result which we will need in the next step.

Let F_0 and F_1 be subsets of an Abelian topological group G such that there exists a symmetric neighborhood W of 0 such that $(F_0+W)\cap F_1=\emptyset$. Then there exists x in $C(G)$ such that $x(g)=-1$ if $g\in F_0$ and $x(g)=1$ if $g\in F_1$.

To prove this we first note that we can assume F_0 and F_1 to be closed, for otherwise we replace F_0 by $\overline{F}_0$, F_1 by $\overline{F}_1$ and W by $W_{1/3}$ where $W_{1/3}$ stands for any symmetric neighborhood of 0 such that $W_{1/3}+W_{1/3}+W_{1/3}\subset W$.

Let now $V_1=F_1^c$. We define $V_{1/2}$ to be $F_0+W_{1/3}$. Then $F_0\subset V_{1/2}\subset \overline{V}_{1/2} \subset V_1$. Since $(F_0+W_{1/3})\cap V_{1/2}^c=\emptyset=(\overline{V}_{1/2}+W_{1/3})\cap F_1$ we can repeat the above process with the pairs of closed sets $(F_0, V_{1/2}^c)$ and $(\overline{V}_{1/2}, F_1)$ to get sets $V_{1/4}$ and $V_{3/4}$. We continue this process to get an open set V_t for each t of the form $(m/2^n)$, $0<t\leqq 1$, such that

$$F_0 \subset V_t, \quad \overline{V}_t \subset V_1 \quad \text{and} \quad \overline{V}_t \subset V_{t'} \quad \text{if } t < t'.$$

We define $y\in C(G)$ as follows:

$$y(g) = \begin{cases} 1 \text{ if } g \notin \bigcup_t V_t, \\ \operatorname{Inf}_{g\in V_t} t \text{ otherwise.} \end{cases}$$

Thus $y(g)=0$ if $g\in F_0$ and $y(g)=1$ if $g\in F_1$. Then $x\in C(G)$ defined by

$$x(g) = 2y(g) - 1$$

satisfies our requirements.

(f) Suppose that

$$\overline{H+V} = G.$$

By means of (ii) of (d) and the result proved in (e) we construct $x_0\in C(G)$

such that x_0 takes the value -1 on $A+V$ and the value 1 on $B+V$. For $x\in C(G)$, let

$$p(x) = \operatorname*{Inf}_{\sigma_1,\cdots,\sigma_n} \operatorname*{Sup}_{\sigma} \frac{1}{n}\sum_{j=1}^{n} x(\sigma_j + \sigma)$$

where the Inf is taken over all finite sequences of elements of G. If

$$\sigma_1, \cdots, \sigma_n \in G = \overline{H+V}$$

we can find g such that $\sigma_1+g, \cdots, \sigma_n+g$ are all in $H+V$. So we may assume that $\sigma_1, \cdots, \sigma_n$ are in $H+V$. Let $\sigma_i=h_i+v_i$, $1\leqq i\leqq n$. It follows from (i) of (d) that $p(x_0)$ and $p(-x_0)$ are both larger than or equal to 1. Thus $p(x_0) \neq -p(-x_0)$ and so by Lemma 1 on page 37 of [2] we see that there are many invariant means on $C(G)$.

(f′) Suppose that

$$\overline{H+V} \neq G.$$

In this case G_2 is an open and closed subgroup of G and so G/G_2 is discrete. If G/G_2 is infinite, there are many invariant means on G/G_2 [1] and hence also on G. So assume that G/G_2 is finite. Let $V_2=V\cap G_2$, $W_2=W\cap G_2$ and $H_2=H\cap G_2$.

If possible let

$$\overline{H_2+V_2} \neq G_2.$$

Then there exists a nonempty set U_2, open in G_2 and so also in G, such that $U_2\cap(H_2+V_2)=\emptyset$. We claim that $U_2\cap(H+V)=\emptyset$ for otherwise there exists $u_2\in U_2$, $h\in H$ and $v\in V$ such that $u_2=h+v$. But then $0=2u_2=2h+2v$. Consequently $2h=2v=0$, i.e., $h\in H_2$ and $v\in V_2$. This however implies that $u_2\in H_2+V_2$ which is not true. Thus our claim is established. Since U_2 is open it follows that

$$U_2 \cap (\overline{H+V}) = \emptyset.$$

This however contradicts (b). Therefore

$$\overline{H_2+V_2}$$

must be equal to G_2.

Since G/G_2 is finite, H_2 must be infinite. Moreover $(V_2+V_2+W_2)\cap H_2$ is finite. Therefore we can construct a function $x_2\in C(G_2)$ for which $p_2(x_2)$ and $p_2(-x_2)$ are $\geqq 1$ where p_2 has the obvious meaning. We now define $x\in C(G)$ by

$$x(\sigma + g_2) = x_2(g_2)$$

where σ belongs to a fixed set of representatives mod G_2. We can easily satisfy ourselves that for this x, $p(x) \neq -p(-x)$.

This completes the proof of the theorem.

Bibliography

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

2. Indar S. Luthar, *Uniqueness of the invariant mean on an Abelian semi-group*, Illinois J. Math. **3** (1959), 28–44.

3. R. J. Silverman, *Invariant linear functions*, Trans. Amer. Math. Soc. **81** (1956), 411–424.

4. André Weil, *L'integration dans les groupes topologiques et ses applications*, Hermann, Paris, 1950.

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